Maximum Correlation Adjustment in Geometrical Deformation Analysis

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Abstract. As an alternative to standard approaches in geometrical deformation analysis a comparison of forms is introduced based on a maximum correlation adjustment. Using this strategy it is possible to obtain correct results for the classification of points into stable and instable even in cases where solutions based on metric criteria fail.

Keywords. Deformation analysis, adjustment, maximum correlation adjustment, Helmert transformation, analysis of residuals

1 Introduction

In almost all geodetic applications the determination of unknown values and the data analysis are done by using a least squares adjustment based on the condition $\sum pvv \rightarrow \min$. It is well-known that this methodology can lead to a correct result only if the functional model is complete up to random errors in the data. In case of an incomplete model the obtained solutions for the unknown values and the adjusted observations can become unrealistic. Moreover, an extension of the model based on an analysis of residuals often becomes difficult or even impossible.

A problem in which it is known in advance that the data does not contain only random effects is the geometrical deformation analysis. Nevertheless, standard solutions are based on a least squares adjustment combined with statistical tests. For details see Pelzer (1971) and Niemeier (1979), an up-to-date overview is offered by Welsch et al. (2000). On the basis of object's coordinates in at least two epochs, the epoch that has to be examined is transformed to a reference epoch using some kind of transformation (e.g. 4-parameter transformation). Afterwards one tries to obtain information about the deformations from the resulting residuals. However, the functional model must be regarded as incomplete because of point shifts having larger amounts than those of random errors. Under such circumstances it is not easy and sometimes even impossible to interpret the residuals because they are composed of three parts:

residual after transformation = real deformation + transformation defect + random error

The only possibility is to classify the points into groups of stable and instable ones. In standard solutions this decision is done using metric criteria for the interpretation of least squares residuals. An example which shows that one of these solutions („localization with S-transformation") may yield a wrong result can be found in Reinking (1994).

As an alternative approach a comparison of forms based on the correlation coefficient is introduced. After a maximum correlation adjustment (MCA[1]) the respective coefficient of correlation can be interpreted as a measure for the extent of similarity between the sets of points in two epochs. This adjustment is justified strictly geometrically and thus a kind of non-probabilistic methodology. Using this strategy it is possible to identify the stable and the instable points even in cases where solutions based on metric criteria fail.

2 Deformation analysis and its fundamental principle

The deformation analysis deals in the first place with the detection of tectonic movements, periodic control surveys of objects and the transformation of heterogeneous geodetic networks into a global datum. In all these cases the fundamental principle is the same. On the basis of repeated measurements the coordinates of two (ore more) epochs are compared. In practice we can divide the problem into two essentially different cases:

1. the geodetic datum is known in advance,
2. the geodetic datum is unknown (general case).

[1] MCA = Maximum Correlation Adjustment

In the first case the residuals are composed only of two parts:

computed deviation = deformation + measurement error

The decision if the deviations should be regarded as deformations is usually obtained by using statistical tests.

When the geodetic datum is unknown a transformation is needed in order to transform the epochs into a common datum. Therefore the transformation parameters have to be determined. This is usually done using a least squares adjustment (e.g. Helmert transformation in a plane). However, the residuals after such a transformation are not easy to interpret and sometimes it is even impossible because they are composed of three parts:

residual after transformation = real deformation + transformation defect + measurement error

The only solution is to classify the points into groups of stable and instable ones but the obtained transformation parameters are not always a suitable base for a decision because of the transformation defect caused by using instable points.

For a comparison of two epochs a lot of methods can be found in the literature. An overview of standard solutions is included in Welsch et al. (2000). All of these are based on analyzing amounts of residuals (e.g. by minimizing them by a least squares adjustment) so that they are all based on *metric criteria*. In the following section an example is given which shows that this procedure doesn't function in every case.

3 Deformation analysis with metric criteria

The following example is taken from Reinking (1994) and it shows that one of the standard solutions is not able to find the congruent group of points in the considered case. For the network shown in Figure 1 the coordinates in two epochs are listed in Table 1.These coordinates are already in a common datum so that we can see that the points 7, 8 and 9 are a congruent group (identical coordinates in both epochs) but the geodetic datum will be regarded as unknown and this information will not be used in the following calculations.

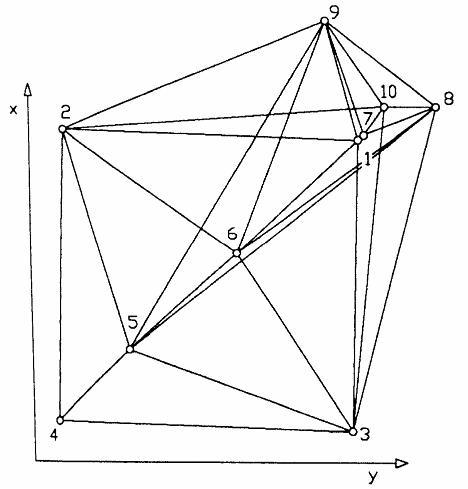

Figure 1. Geodetic network

Table 1. Coordinates of the network points

	Epoch I		Epoch II	
Point No.	Y [m]	X [m]	y [m]	x [m]
1	220.00	220.00	222.00	217.50
2	20.00	220.00	22.50	222.50
3	220.00	20.00	217.50	17.50
4	20.00	20.00	16.00	25.50
5	70.00	70.00	68.00	73.00
6	140.00	140.00	140.00	140.50
7	225.00	220.00	225.00	220.00
8	275.00	240.00	275.00	240.00
9	200.00	300.00	200.00	300.00
10	240.00	240.00	242.00	237.50

In this case some kind of transformation has to be applied and Reinking (1994) used a method called „localization with S-transformation“ with a fixed scale between both epochs. The value of the test quantity R_f was calculated, for details see Niemeier (1985) and the point associated with the lowest value of R_f (see Table 2) is declared to be instable.

Table 2. Test quantity R_f

Point	R_f	Point	R_f
1	14.16	6	14.63
2	13.60	7	14.68
3	12.92	8	12.69
4	12.74	9	12.67
5	15.50	10	13.74

In this example point 9 is associated with the lowest value of R_f so that this point should be eliminated from the group of stable points. However, we know that point 9 is stable and so this method cannot lead to a correct result in this example.

4 Comparison of forms using maximum correlation adjustment

As an alternative to the analysis based on metric criteria a *comparison of forms* will be introduced. This comparison can be based on similarity which is described by the correlation coefficient (squared) r^2.

4.1 The correlation coefficient

The well-known definition for the correlation coefficient between two sets of real numbers (one-dimensional case) is

$$r^2 = \frac{\left(\sum_{i=1}^{n}(w_i - \overline{w})(z_i - \overline{z})\right)^2}{\sum_{i=1}^{n}(w_i - \overline{w})^2 \sum_{i=1}^{n}(z_i - \overline{z})^2} \tag{1}$$

with

$$w_i \in \Re, z_i \in \Re \quad (i = 1, \dots, n) \tag{2}$$

and

$$\overline{w} = \frac{1}{n}\sum_{i=1}^{n} w_i \ , \ \overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i \ . \tag{3}$$

The points of a geodetic network are not just real numbers, therefore we have to define the correlation coefficient in a generalized form. For the comparison of forms we have n quantities in each epoch which can be regarded as vectors in a m-dimensional vector space (e.g. m = 2 in a plane). The vectors

$$\mathbf{w}_i = \left[w_{i_1}, w_{i_2}, \dots, w_{i_m}\right]^{\mathrm{T}} \in \Re^m, (i = 1, \dots, n) \tag{4}$$

and

$$\mathbf{z}_i = \left[z_{i_1}, z_{i_2}, \dots, z_{i_m}\right]^{\mathrm{T}} \in \Re^m, (i = 1, \dots, n) \tag{5}$$

contain the coordinates of points in epoch 1 resp. 2. Now it is possible to take the formula for real numbers but to replace the multiplication of real numbers with the inner product of vectors and to obtain

$$r^2 = \frac{\left(\sum_{i=1}^{n}(\mathbf{w}_i - \overline{\mathbf{w}})\cdot(\mathbf{z}_i - \overline{\mathbf{z}})\right)^2}{\sum_{i=1}^{n}(\mathbf{w}_i - \overline{\mathbf{w}})^2 \sum_{i=1}^{n}(\mathbf{z}_i - \overline{\mathbf{z}})^2} \tag{6}$$

with

$$\overline{\mathbf{w}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{w}_i \ \text{ and } \ \overline{\mathbf{z}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \ . \tag{7}$$

4.2 Maximum correlation adjustment (MCA)

The amount of the correlation coefficient r^2 is a consequence of

1. the extent of the similarity of forms,
2. the relative position of the configurations.

This is illustrated on a simple geometrical example with two similar triangles. In order to compute a correlation coefficient we have to use coordinates as it is visible from the definition (6). Hence we must choose some coordinate systems. In the relative position shown in Figure 2 the result of the computed correlation coefficient is r^2 = 1. If we choose the coordinate systems in such a way that we get the relative position of the triangles shown in Figure 3 the correlation coefficient will no longer be r^2 = 1.

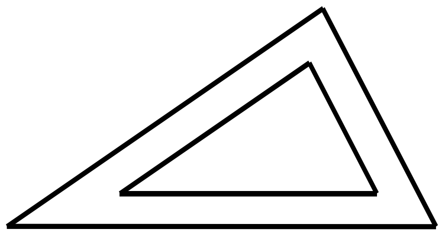

Figure 2. Two similar triangles in a homothetic position

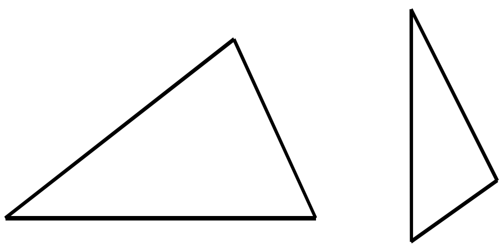

Figure 3. Two similar triangles in an arbitrary position

Therefore, in order to determine only the extent of similarity we must consider all possible positions of both coordinate systems and choose the one which yields the maximum correlation coefficient. This is done by a maximum correlation adjustment. In such a way we get the correlation coefficient r^2 which is free from influence of the coordinate systems and describes only the extent of similarity.

4.3 Definition and characteristics of a maximum correlation adjustment

The solution of a maximum correlation adjustment was defined by Petrovic (1997):

> Let $\boldsymbol{F}$ be a given class of real functions. If there exists an element $f \in \boldsymbol{F}$ such that the correlation coefficient (squared) r^2 $(l, f(x))$ obtains the maximal possible value with respect to all elements of the class $\boldsymbol{F}$, then the function f is the solution of the problem of maximum correlation adjustment.

This adjustment has the following property, see Petrovic (1991):

> If $f \in \boldsymbol{F}$ is a solution by maximum correlation and for $c_1, c_2 \in \Re$ it holds $c_1 + c_2 f \in \boldsymbol{F}$, then $c_1 + c_2 f$ is also a solution, namely $r^2(\ell, c_1 + c_2 f) = r^2(\ell, f)$.

With this property the solution of a maximum correlation adjustment is generally not unique but a whole class. From this class we have to choose one suitable solution.

5 Maximum correlation adjustment between two-dimensional position vectors

5.1 Functional model

For homologous points the vectors

$$[X, Y, 0]^{\mathrm{T}}[\vec{e}_i]_z \text{ and } [x, y, 0]^{\mathrm{T}}[\vec{e}_i]_w \tag{8}$$

are given. Both bases $[\vec{e}_i]_z$ and $[\vec{e}_i]_w$ define a geodetic coordinate system, whereby the second basis can be transformed onto the first one by a 4-parameter-transformation. Using the parameters

X_0, Y_0	translation of the origin,
α	rotation angle,
m	scale factor

we can formulate the transformation of bases

$$[\vec{e}_i]_z = [X_0, Y_0, 0]^{\mathrm{T}} + m\,[R_3(\alpha)]^{\mathrm{T}}\,[\vec{e}_i]_w\ . \tag{9}$$

Thus we obtain also the transformation for the vector components

$$\begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ 0 \end{bmatrix} \tag{10}$$

and in expanded form

$$X = (m\cos\alpha)\,x - (m\sin\alpha)\,y + X_0$$

and

$$Y = (m\sin\alpha)\,x + (m\cos\alpha)\,y + Y_0\ . \tag{11}$$

Introducing the abbreviations

$$a = m\cos\alpha \text{ and } o = m\sin\alpha \tag{12}$$

yields

$$X = a\,x - o\,y + X_0 \text{ and } Y = o\,x + a\,y + Y_0 \tag{13}$$

for the transformation formulas.

5.2 The correlation coefficient

On the basis of the equations (13) the correlation coefficient between the vectors

$$\mathbf{w}_i = \begin{bmatrix} w_{i_x} \\ w_{i_y} \end{bmatrix} = \begin{bmatrix} a\,x_i - o\,y_i + X_0 \\ a\,y_i + o\,x_i + Y_0 \end{bmatrix}$$

and

$$\mathbf{z}_i = \begin{bmatrix} z_{i_x} \\ z_{i_y} \end{bmatrix} = \begin{bmatrix} X_i \\ Y_i \end{bmatrix} \tag{14}$$

can be established. For the mean values $\overline{\mathbf{w}}$ and $\overline{\mathbf{z}}$ we obtain (with n = number of homologous points)

$$\begin{aligned} \overline{\mathbf{w}} = \tfrac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i = \begin{bmatrix} \overline{w}_x \\ \overline{w}_y \end{bmatrix} &= \begin{bmatrix} \tfrac{1}{n}\sum_{i=1}^{n} w_{i_x} \\ \tfrac{1}{n}\sum_{i=1}^{n} w_{i_y} \end{bmatrix} \\ &= \frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n}(a\,x_i - o\,y_i + X_0) \\ \sum_{i=1}^{n}(a\,y_i + o\,x_i + Y_0) \end{bmatrix} \\ &= \frac{1}{n}\begin{bmatrix} a\sum_{i=1}^{n} x_i - o\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} X_0 \\ a\sum_{i=1}^{n} y_i + o\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} Y_0 \end{bmatrix} \end{aligned} \tag{15}$$

with

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \quad (16)$$

respectively

$$\bar{\mathbf{w}} = \frac{1}{n}\begin{bmatrix} n\,a\,\bar{x} - n\,o\,\bar{y} + n\,X_0 \\ n\,a\,\bar{y} + n\,o\,\bar{x} + n\,Y_0 \end{bmatrix} = \begin{bmatrix} a\,\bar{x} - o\,\bar{y} + X_0 \\ a\,\bar{y} + o\,\bar{x} + Y_0 \end{bmatrix} \quad (17)$$

and in a similar way

$$\bar{\mathbf{z}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i = \begin{bmatrix} \bar{z}_x \\ \bar{z}_y \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} z_{i_x} \\ \frac{1}{n}\sum_{i=1}^{n} z_{i_y} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} Y_i \end{bmatrix} \quad (18)$$

with

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \quad (19)$$

$$\bar{\mathbf{z}} = \frac{1}{n}\begin{bmatrix} n\,\bar{X} \\ n\,\bar{Y} \end{bmatrix} = \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}. \quad (20)$$

Using the vectors (14), (17) and (20) the correlation coefficient (6) yields

$$r^2 = \left\{ \sum_{i=1}^{n} \left(\begin{bmatrix} a\,x_i - o\,y_i + X_0 \\ a\,y_i + o\,x_i + Y_0 \end{bmatrix} - \begin{bmatrix} a\,\bar{x} - o\,\bar{y} + X_0 \\ a\,\bar{y} + o\,\bar{x} + Y_0 \end{bmatrix} \right) \left(\begin{bmatrix} X_i \\ Y_i \end{bmatrix} - \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} \right) \right\}^2 \Bigg/ \left\{ \sum_{i=1}^{n} \left(\begin{bmatrix} a\,x_i - o\,y_i + X_0 \\ a\,y_i + o\,x_i + Y_0 \end{bmatrix} - \begin{bmatrix} a\,\bar{x} - o\,\bar{y} + X_0 \\ a\,\bar{y} + o\,\bar{x} + Y_0 \end{bmatrix} \right)^2 \sum_{i=1}^{n} \left(\begin{bmatrix} X_i \\ Y_i \end{bmatrix} - \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} \right)^2 \right\} . \quad (21)$$

After forming the respective inner products and some simplifications we obtain

$$r^2 = \left\{ m^2 \left(\cos\alpha \left(\sum_{i=1}^{n} x_i X_i - n\bar{x}\bar{X} + \sum_{i=1}^{n} y_i Y_i - n\bar{y}\bar{Y} \right) + \sin\alpha \left(\sum_{i=1}^{n} x_i Y_i - n\bar{x}\bar{Y} - \sum_{i=1}^{n} y_i X_i + n\bar{y}\bar{X} \right) \right)^2 \right\} \Bigg/ \left\{ m^2 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \right) \times \left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2 + \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2 \right) \right\} . \quad (22)$$

It is visible that the scale factor m can be eliminated so that the expression obtains its final form

$$r^2 = \left\{ \left(\cos\alpha \left(\sum_{i=1}^{n} x_i X_i - n\bar{x}\bar{X} + \sum_{i=1}^{n} y_i Y_i - n\bar{y}\bar{Y} \right) + \sin\alpha \left(\sum_{i=1}^{n} x_i Y_i - n\bar{x}\bar{Y} - \sum_{i=1}^{n} y_i X_i + n\bar{y}\bar{X} \right) \right)^2 \right\} \Bigg/ \left\{ \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \right) \times \left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2 + \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2 \right) \right\} . \quad (23)$$

The proven properties of this expression for the correlation coefficient are formulated in the following theorem.

Theorem 1: *Let n > 2 homologous points be given in coordinate systems that can be transformed one onto another by a 4-parameter transformation and let the coordinates be regarded as position vectors in a two-dimensional vector space. Then it holds that the correlation coefficient is independent from the translation parameters X_0 and Y_0 and from the scale factor m.*

5.3 Searching for maxima

In a maximum correlation adjustment the solution is to be determined in such a way that the correlation coefficient (23) obtains its maximal possible value, i.e. $r^2 \rightarrow \max \leq 1$. In order to determine this extreme value we assume that it is associated to an interior point of the domain of definition of the function r^2. Thus we have to find a relative maximum. With $r^2 = r^2(\alpha)$ the correlation coefficient (23) is a function of one variable and the stationary points can be determined by solving the equation $dr^2/d\alpha = 0$. According to Bronstein et al. (1995) we can determine the character of each stationary point using

1. the method of sign comparison, or
2. the method of the higher derivatives.

5.4 Determination of the derivative dr^2/dα

With the abbreviations

$$
\begin{aligned}
A &= \sum_{i=1}^{n} x_i X_i - n\bar{x}\bar{X} + \sum_{i=1}^{n} y_i Y_i - n\bar{y}\bar{Y} \\
B &= \sum_{i=1}^{n} x_i Y_i - n\bar{x}\bar{Y} - \sum_{i=1}^{n} y_i X_i + n\bar{y}\bar{X} \\
C_1 &= \sum_{i=1}^{n} x_i^{\,2} - n\bar{x}^2 + \sum_{i=1}^{n} y_i^{\,2} - n\bar{y}^2 \\
C_2 &= \sum_{i=1}^{n} X_i^{\,2} - n\bar{X}^2 + \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2
\end{aligned}
\tag{24}
$$

and

$$C = C_1 C_2$$

we obtain from (23) the following expression for the correlation coefficient

$$r = \sqrt{\frac{(A\cos\alpha + B\sin\alpha)^2}{C}} = \pm\frac{A\cos\alpha + B\sin\alpha}{\sqrt{C}} \ . \tag{25}$$

Taking expression (25) as the function to be differentiated yields

$$\frac{d\,r}{d\,\alpha} = \pm\frac{B\cos\alpha - A\sin\alpha}{\sqrt{C}} \ . \tag{26}$$

Hence we can solve the equation dr/dα = 0. On the basis of

$$0 = B\cos\alpha - A\sin\alpha \tag{27}$$

we obtain with $A \neq 0$ and $\cos\alpha \neq 0$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{B}{A} \ . \tag{28}$$

Substituting for the abbreviations (24) yields

$$\alpha = \arctan\frac{\sum_{i=1}^{n} x_i Y_i - n\bar{x}\bar{Y} - \sum_{i=1}^{n} y_i X_i + n\bar{y}\bar{X}}{\sum_{i=1}^{n} x_i X_i - n\bar{x}\bar{X} + \sum_{i=1}^{n} y_i Y_i - n\bar{y}\bar{Y}} \ . \tag{29}$$

The relationship between this rotation angle obtained from a maximum correlation adjustment and the result from a Helmert transformation will be considered in section 6.

5.5 Analysis of residuals

After determining the rotation angle α by solving equation (29) we can use this solution and the free parameters m, X_0 and Y_0 to represent the adjusted function values. Thus we can establish the vector of residuals $\mathbf{v}_i$ for every point P_i in the form

$$\mathbf{v}_i = \begin{bmatrix} (m\cos\alpha)\, x_i - (m\sin\alpha)\, y_i + X_0 \\ (m\cos\alpha)\, y_i + (m\sin\alpha)\, x_i + Y_0 \end{bmatrix} - \begin{bmatrix} X_i \\ Y_i \end{bmatrix} \tag{30}$$

with

$m \in \Re^+$ freely selectable,
$X_0, Y_0 \in \Re$ freely selectable.

Since we can generate a whole class of solutions using the free parameters, we obtain a whole class of vectors of residuals for each point. These vectors, resp. the graphic representation of them have to be interpreted in order to determine systematic point shifts. For a structured data analysis it is useful to decompose the vectors of residuals into the components v_X and v_Y and to establish the vectors

$$\mathbf{v}_X = \begin{bmatrix} (m\cos\alpha)\, x_1 - (m\sin\alpha)\, y_1 + X_0 \\ \vdots \\ (m\cos\alpha)\, x_n + (m\sin\alpha)\, y_n + X_0 \end{bmatrix} - \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \tag{31}$$

and

$$\mathbf{v}_Y = \begin{bmatrix} (m\cos\alpha)\, y_1 + (m\sin\alpha)\, x_1 + Y_0 \\ \vdots \\ (m\cos\alpha)\, y_n + (m\sin\alpha)\, x_n + Y_0 \end{bmatrix} - \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} . \tag{32}$$

From these vectors it is visible that the parameters X_0 and Y_0 don't yield any gain of information, because they do not influence relative relations between the respective residuals. For this reason a variation of these parameters is not meaningful. Therefore we will take

X_0 = const. and Y_0 = const.

and generate the class of vectors $\mathbf{v}_X$ and $\mathbf{v}_Y$ only by variation of the parameter m. In order to choose *one* solution (out of the class) we need some hypothesis regarding the behaviour of the residuals, e.g.:

- smoothness of the residuals within groups of points,
- certain amounts for the residuals of groups of points.

After that an identification of systematic influences or the classification of points into groups with different deformations follows.

6 Relations between Helmert transformation and MCA with two-dimensional position vectors

Numerical investigations have shown that the rotation angle obtained from a Helmert transformation corresponds to the solution for a 4-parameter transformation by maximum correlation adjustment. In the following it will be shown that this holds in general.

6.1 Proof that the Helmert solution is one of the MCA solutions

On the basis of the solution for a Helmert transformation that can be found e.g. in Großmann (1969), the formula for the rotation angle[2] is

$$\alpha = \arctan \frac{\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)\left(x_i-\overline{x}\right)-\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(y_i-\overline{y}\right)}{\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(x_i-\overline{x}\right)+\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)\left(y_i-\overline{y}\right)} \;. \quad (33)$$

In the following proof it is to be shown that the solution for the rotation angle in equation (33) is equal to the one in equation (29), i.e.

$$\arctan \frac{\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)\left(x_i-\overline{x}\right)-\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(y_i-\overline{y}\right)}{\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(x_i-\overline{x}\right)+\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)\left(y_i-\overline{y}\right)} = \arctan \frac{\sum_{i=1}^{n} x_i Y_i - n\overline{x}\overline{Y} - \sum_{i=1}^{n} y_i X_i + n\overline{y}\overline{X}}{\sum_{i=1}^{n} x_i X_i - n\overline{x}\overline{X} + \sum_{i=1}^{n} y_i Y_i - n\overline{y}\overline{Y}} \;. \quad (34)$$

Proof:

The right-hand side of equation (34) can be brought in the following form

$$\arctan \frac{\left(\sum_{i=1}^{n} x_i Y_i - n\overline{x}\overline{Y} - n\overline{x}\overline{Y} + n\overline{x}\overline{Y} - \sum_{i=1}^{n} y_i X_i + n\overline{y}\overline{X} + n\overline{y}\overline{X} - n\overline{y}\overline{X}\right)}{\left(\sum_{i=1}^{n} x_i X_i - n\overline{x}\overline{X} - n\overline{x}\overline{X} + n\overline{x}\overline{X} + \sum_{i=1}^{n} y_i Y_i - n\overline{y}\overline{Y} - n\overline{y}\overline{Y} + n\overline{y}\overline{Y}\right)} \;. \quad (35)$$

Substituting (16) and (19) yields

$$\arctan \frac{\left\{\sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \overline{x} Y_i - \sum_{i=1}^{n} x_i \overline{Y} + \sum_{i=1}^{n} \overline{x}\overline{Y} - \left(\sum_{i=1}^{n} y_i X_i - \sum_{i=1}^{n} \overline{y} X_i - \sum_{i=1}^{n} y_i \overline{X} + \sum_{i=1}^{n} \overline{y}\overline{X}\right)\right\}}{\left\{\sum_{i=1}^{n} x_i X_i - \sum_{i=1}^{n} \overline{x} X_i - \sum_{i=1}^{n} x_i \overline{X} + \sum_{i=1}^{n} \overline{x}\overline{X} + \sum_{i=1}^{n} y_i Y_i - \sum_{i=1}^{n} \overline{y} Y_i - \sum_{i=1}^{n} y_i \overline{Y} + \sum_{i=1}^{n} \overline{y}\overline{Y}\right\}} \;. \quad (36)$$

It follows

$$\arctan \frac{\left\{\sum_{i=1}^{n}\left(x_i Y_i - \overline{x} Y_i - x_i \overline{Y} + \overline{x}\overline{Y}\right) - \sum_{i=1}^{n}\left(y_i X_i - \overline{y} X_i - y_i \overline{X} + \overline{y}\overline{X}\right)\right\}}{\left\{\sum_{i=1}^{n}\left(x_i X_i - \overline{x} X_i - x_i \overline{X} + \overline{x}\overline{X}\right) + \sum_{i=1}^{n}\left(y_i Y - \overline{y} Y_i - y_i \overline{Y} + \overline{y}\overline{Y}\right)\right\}} \quad (37)$$

and finally

$$\arctan \frac{\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)\left(x_i-\overline{x}\right)-\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(y_i-\overline{y}\right)}{\sum_{i=1}^{n}\left(X_i-\overline{X}\right)\left(x_i-\overline{x}\right)+\sum_{i=1}^{n}\left(Y_i-\overline{Y}\right)\left(y_i-\overline{y}\right)} \;. \quad (38)$$

The expression (38) is equal to the expression on the left-hand side in equation (34), which had to be demonstrated. Hence the following theorem has been proved:

Theorem 2: *Let $n > 2$ homologous points be given in coordinate systems that can be transformed one onto another by a 4-parameter transformation and let the coordinates be regarded as position vectors in a two-dimensional vector space. Then it holds that the rotation angle α in the solution of a maximum correlation adjustment is identical to the rotation angle obtained from a Helmert transformation.*

The conclusion of theorems 1 and 2 is that the Helmert transformation is to be regarded as a special case of the respective maximum correlation adjustment.

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[2] The notation has been adapted to the one used in this article.

6.2 Generating the class of all MCA solutions from the Helmert solution

In section 6.1 it was shown that the solution for a Helmert transformation is *one* of the solutions for the respective maximum correlation adjustment. This relationship will now be considered more detailed using the following theorem that can be found in Petrovic (1997):

If the solution by maximum correlation $f\,(x)$ is searched for in a class $\boldsymbol{F}$ of functions which has the property

$$\forall\, f\,(x) \in \boldsymbol{F} \quad \& \quad \forall\, c_1, c_2 \in \Re$$
$$\Rightarrow\; c_1 + c_2\, f\,(x) \in \boldsymbol{F}, \qquad (39)$$

then it holds:

the solution by maximum correlation is not unique, but a whole class

$$\{f \mid f\,(x) = c_1 + c_2\, f_0\,(x),\; c_1, c_2 \in \Re\,\} \subseteq \boldsymbol{F}, \qquad (40)$$

where the base function $f_0\,(x)$ can be any of the solutions.

Therefore, if the class $\boldsymbol{F}$ of all 4-parameter transformations has the property (39) the class of all solutions by maximum correlation can be generated from the result of the respective least squares adjustment. For the functions of a 4-parameter transformation (11) the requirement (39) yields

$$c_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} (m\cos\alpha)\, x_1 - (m\sin\alpha)\, y_1 + X_0 \\ (m\cos\alpha)\, y_1 + (m\sin\alpha)\, x_1 + Y_0 \\ \vdots \\ (m\cos\alpha)\, x_n - (m\sin\alpha)\, y_n + X_0 \\ (m\cos\alpha)\, y_n + (m\sin\alpha)\, x_n + Y_0 \end{bmatrix} \in \boldsymbol{F}. \quad (41)$$

Proof:

The expression (41) can be brought in the following form

$$\begin{bmatrix} (c_2 m\cos\alpha)\, x_1 - (c_2 m\sin\alpha)\, y_1 + c_2 X_0 + c_1 \\ (c_2 m\cos\alpha)\, y_1 + (c_2 m\sin\alpha)\, x_1 + c_2 Y_0 + c_1 \\ \vdots \\ (c_2 m\cos\alpha)\, x_1 - (c_2 m\sin\alpha)\, y_1 + c_2 X_0 + c_1 \\ (c_2 m\cos\alpha)\, y_1 + (c_2 m\sin\alpha)\, x_1 + c_2 Y_0 + c_1 \end{bmatrix}. \quad (42)$$

Substituting

$$m' = c_2 m\,,\; X_0' = c_2 X_0 + c_1$$

and

$$Y_0' = c_2 Y_0 + c_1 \qquad (43)$$

yields

$$\begin{bmatrix} (m'\cos\alpha)\, x_1 - (m'\sin\alpha)\, y_1 + X_0' \\ (m'\cos\alpha)\, y_1 + (m'\sin\alpha)\, x_1 + Y_0' \\ \vdots \\ (m'\cos\alpha)\, x_n - (m'\sin\alpha)\, y_n + X_0' \\ (m'\cos\alpha)\, y_n + (m'\sin\alpha)\, x_n + Y_0' \end{bmatrix} \in \boldsymbol{F}\,. \qquad (44)$$

This expression belongs to the same class of functions $\boldsymbol{F}$ as the original function of a 4-parameter transformation (11), which had to be demonstrated. Therefore it is proved that the class of all solutions by maximum correlation can be generated from the result of a Helmert transformation. It is to remark that instead of the proof in section 6.1 the following theorem from Petrovic (1991) can be used.

If $\boldsymbol{F}$ has the property (39) the solution by least squares is included in the subclass (40).

The investigations in section 6.1 and 6.2 can be summarized in the following theorem:

Theorem 3: *Let $n > 2$ homologous points be given in coordinate systems that can be transformed one onto another by a 4-parameter transformation and let the coordinates be regarded as position vectors in a two-dimensional vector space. Then it holds that the result of a Helmert transformation is one of the solutions out of the class of all solutions by maximum correlation. Furthermore, the class of all solutions by maximum correlation can be generated from the result of the Helmert transformation.*

6.3 Importance for the application in practice

The theorem 3 is of great importance for application in practice. The class of all solutions by maximum correlation can be generated from the result of a Helmert transformation, so that available standard software can be used to obtain this base function. In Figure 4 this strategy is shown in a flowchart.

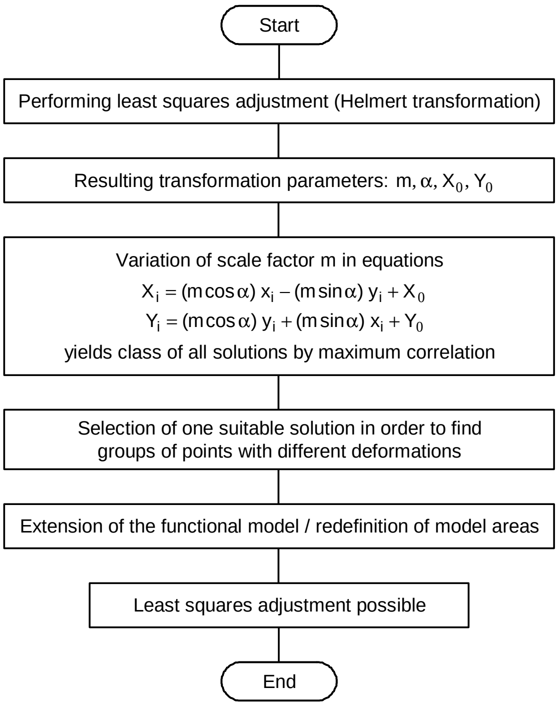

Figure 4. Generating the class of all MCA solutions from the Helmert solution

6.4 Deformation analysis based on comparison of forms

As an example the same network from section 3 is used. Choosing the solution α = 398.7078, m = 1.0148, X_0 = -5.800, Y_0 = arbitrary from the class of all solutions by MCA we obtain the residuals v_X shown in Figure 5.

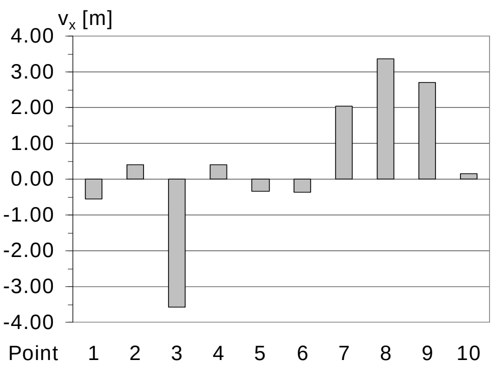

Figure 5. Residuals v_X for the chosen MCA solution

The solution α = 398.7078, m = 0.9917, X_0 = arbitrary, Y_0 = 4.350 yields the residuals v_Y shown in Figure 6.

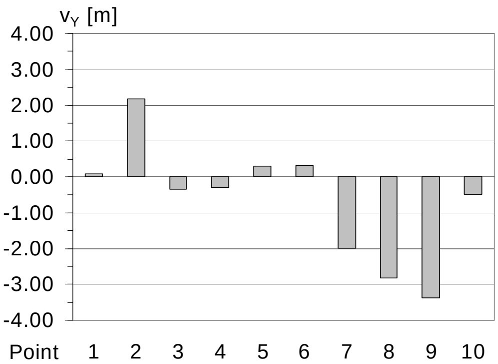

Figure 6. Residuals v_Y for the chosen MCA solution

The interpretation of both sets of residuals is that the points can be classified into two groups:

- group 1 consisting of the points 1, 2, 3, 4, 5, 6, 10,
- group 2 consisting of the points 7, 8, 9.

After this classification the deformation analysis can continue. For example, a least squares adjustment for each group can be performed with the result that group 1 is to be regarded as instable and group 2 as the group of stable points. This result is in accordance with the real deformations that were known in advance in this example.

7 Conclusion

In some cases an application of metric criteria in geometrical deformation analysis results in a wrong identification of stable points. As an alternative to standard solutions a comparison of forms based on a maximum correlation adjustment is introduced. This adjustment is justified strictly geometrically and thus a kind of non-probabilistic methodology. Using this strategy the computation of more realistic residuals is demonstrated on a numerical example on which metric criteria failed.

Moreover, fundamental relations between Helmert transformation and the respective maximum correlation adjustment have been shown.

References

Bronstein, I.N., K.A. Semendjajew, G. Musiol, H. Mühlig (1995). Taschenbuch der Mathematik. *Verlag Harri Deutsch, Thun, Frankfurt am Main*, 2. Auflage

Großmann, W. (1969). Grundzüge der Ausgleichungsrechnung. *Springer-Verlag, Berlin, Heidelberg*, 3. Auflage

Niemeier, W. (1979). Zur Kongruenz mehrfach beobachteter geodätischer Netze. *Wissenschaftliche Arbeiten der Fachrichtung Vermessungswesen der Universität Hannover*, Nr. 88, Hannover

Niemeier, W. (1985). Deformationsanalyse. In: Geodätische Netze in Landes- und Ingenieurvermessung II, Hrsg. Pelzer, H., *Verlag Konrad Wittwer, Stuttgart*

Pelzer, H. (1971). Zur Analyse geodätischer Deformationsmessungen. *DGK, Reihe C*, Nr. 164, München

Petrovic, S. (1991). Geometry of the Correlation Coefficient and its Application in Geodesy. *Mitteilungen der geodätischen Institute der Technischen Universität Graz*, Folge 71

Petrovic, S. (1997). Maximum Correlation Adjustment and Incomplete Models. In: *Proceedings of the Second Turkish-German Joint Geodetic Days*, Berlin, pp. 687-694

Reinking, J. (1994). Geodätische Analyse inhomogener Deformationen mit nichtlinearen Transformationsfunktionen. *DGK, Reihe C*, Nr. 413, München

Welsch, W., O. Heunecke, H. Kuhlmann (2000). Auswertung geodätischer Überwachungsmessungen. *Herbert Wichmann Verlag, Heidelberg*